UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH3506**

ASSESSMENT PATTERN : **MATH3506A**

MODULE NAME : **Mathematical Ecology**

DATE : **21-May-10**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2009/10-MATH3506A-001-EXAM-6

TURN OVER

All questions may be attempted but only marks obtained on the best four *solutions will count.*
The use of an electronic calculator is not *permitted in this examination.*

1. Two interacting species with densities x, y are modelled by the system

$$\begin{aligned} \frac{dx}{dt} &= x(a - bx - cy) \\ \frac{dy}{dt} &= y(-d + ex - fy) \end{aligned} \qquad (1)$$

where $a, b, c, d, e, f > 0$.

   (a) Briefly discuss the model, identifying the type of species-species interactions involved.

   (b) Find all steady states of the system (1) and determine whether they are locally stable or unstable.

   (c) Sketch the phase planes for the system (1) when (i) $ae < bd$ and (ii) $ae > bd$.

2. Consider the following model giving the size, $N_{k+1} \geqslant 0$, of a population at time $k+1$ in terms of the population size at time k:

$$N_{k+1} = rN_k \exp\left(1 - \frac{N_k}{K}\right) = f(N_k), \quad r, K > 0, \quad k = 0, 1, \ldots. \qquad (2)$$

   (a) Briefly discuss possible ecological reasons for choosing a discrete time model in preference to a continuous time model.

   (b) Show that equation (2) can have two steady states, giving the conditions on r for their existence.

   (c) Analyse the local stability of each steady state.

   (d) Sketch a cobweb map (i.e. iterates of N_k versus N_{k+1}) in the cases (i) $1 \leqslant r < e$, and (ii) $\frac{1}{e} < r < 1$.

   (e) What happens when r is just greater than e?

3. A population is modelled by the differential equation

$$\frac{dN}{dt} = \rho N\left(1 - \frac{N}{K(t)}\right), \quad N(0) = N_0, \tag{3}$$

where $\rho > 0$ and $N_0 > 0$ are constants and $K(t) > 0$ is a periodic function with period T.

(a) Give an ecological interpretation of ρ and $K(t)$.

(b) By considering $M(t) = N(t)\, e^{-\rho t}$, or otherwise, show that (3) has the solution

$$N(t) = \frac{N_0 e^{\rho t}}{1 + N_0 G(t)}, \qquad \text{where } G(t) = \int_0^t \frac{\rho}{K(u)} e^{\rho u}\, du.$$

(c) Show that for integer $k \geqslant 0$ and $s \in [0, T)$,

$$G(kT + s) = \left(\frac{1 - e^{k\rho T}}{1 - e^{\rho T}}\right) G(T) + e^{k\rho T} G(s).$$

(d) Show that $N_\infty(s) = \lim_{k\to\infty} N(kT + s)$ is periodic.

4. A predator-prey model has the form

$$\begin{aligned} \frac{dX}{dt} &= \rho X\left(1 - \frac{X}{K}\right) - \phi(X, Y) \\ \frac{dY}{dt} &= Y(\sigma X - \mu), \end{aligned} \tag{4}$$

where $\phi(X, Y) = \dfrac{\gamma X Y}{A + X}$ and $\rho, K, \gamma, A, \sigma$ are all positive constants.

(a) Which of X, Y represents the predator, and which the prey?

(b) Sketch $\phi(X, Y)$ for fixed $Y > 0$. What does the parameter γ represent?

(c) Find all steady states of (4) and determine whether they are locally stable or unstable.

(d) Show that when $K\sigma > 2\mu$ a limit cycle is possible around the interior steady state as A varies and find the critical value A_c of A at which it occurs.

(e) Sketch the phase plane for (4) for A just less than A_c.

5. In an age-structured population there are n age classes. The population density of age k at time t is denoted by $N_k(t)$ and $\mathrm{N}(t) = (N_1(t), ..., N_n(t))^T$. The expected number of offspring of an individual at age k is b_k $(k \geqslant 1)$ and the probability that an individual aged $k \geqslant 0$ (with $k = 0$ the newborns) survives to age $k+1$ is p_k. No individual can survive past age n.

   (a) Show that $\mathrm{N}(t+1) = L\mathrm{N}(t)$ for $t = 0, 1, 2, ...$ where L is a real $n \times n$ matrix which you should find.

   (b) Show that the eigenvalues λ of L satisfy

   $$\sum_{r=1}^{n} \frac{b_r \ell_r}{\lambda^r} = 1,$$

   where $\ell_k = \Pi_{i=0}^{k-1} p_k$ for $1 \leqslant k \leqslant n$. Give an interpretation of each ℓ_k.

   (c) Show that L has a unique positive eigenvalue λ_0.

   (d) Suppose that L is aperiodic and has a complete set of eigenvectors $\{\mathrm{v}_0, \ldots, \mathrm{v}_{n-1}\}$, where v_0 is an eigenvector associated with the unique positive eigenvalue λ_0 of part (c). Show that if $\mathrm{N}(0) = \sum_{i=0}^{n-1} c_i \mathrm{v}_i$ where $c_0 \neq 0$, then for large t each age class grows at rate λ_0.

   (e) If, in addition to the conditions on L specified in (d), you are told that $\sum_{r=1}^{n} b_r \ell_r = 1$, what happens to $\mathrm{N}(t)$ for large t?